UGC Approved Journal

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

ISO 3297:2007 Certified

Vol. 4, Issue 9, September 2017

Public Good Index for Fuzzy Simple Games in Multilinear Extension Form

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Abstract: In this paper, I will mainly concerned on influence of a voter if he shift his rule from a winning situation to no participation (rights to reject) in a decision making situation. I will try to measure the power of voter mathematically using fuzzy approach while he shifts his rule from a winning situation. I will propose public good index in fuzzy version as a method to measure power of voter. I propose the fuzzy versions of Public Good Index (PGI) and corresponding characterizations for simple games with fuzzy coalition (fuzzy game). Finally in this paper Public Good Index is formulated for the class of simple games with fuzzy coalition in multilinear extension form.

Keywords: Decision making, Simple games, player, Fuzzy simple games, Public Good index, fuzzy coalition.

1. INTRODUCTION

Power indices measures the influential factor of a person (player) while he shift his role in decision making situation. In a simple game the value of a winning coalition and loosing coalition can be assessed by binary terms (1 for winning coalition and 0 for loosing coalition). The PGI is important power indices introduced by Holler [1]. PGI measure in a way, the influence of a player while he shift his loyalty from the minimal winning coalitions. A winning coalition said to be a minimal winning coalition if removal of any player makes it into a losing coalition. Here we will use the term voter to indicate the members of minimal winning coalition. The PGI can be determined by the number of minimal winning coalitions containing the voters divided by the sum of such numbers across all the voters, see [1] for a detailed reference.. In this paper we introduce PGI for a fuzzy simple game and obtain the respective characterizations In this paper we will finally introduce PGI for the class of fuzzy simple games in multilinear extension form. Here we follow the characterizations of the PGI due to [1, 8].

The rest of the paper is organized as follows. Section 2 presents the preliminary concepts about the PGI in crisp settings and fuzzy simple games in multilinear extension form . In Section 3, the notions of fuzzy PGI is introduced for fuzzy simple games. Section 4 brings out fuzzy PGI for the class of fuzzy simple games in multilinear extension form followed by an illustrative example. Finally, in Section 5 we present some concluding remarks.

2. PRELIMINARIES

Here we mention some necessary preliminaries concepts relating to cooperative games in crisp and PGI in crisp settings which bear similar notions from [8]. In this section we also mention some notions of fuzzy simple games in multilinear extension form.

2.1 PUBLIC GOOD INDEX

In the following we define the Public Good index for the class G^N as follows.

Definition 2.1. A function $f : G^N \to \mathbb{R}^n$ defines a value due to [8] if for every $S \in 2^N$ it satisfies the following axioms.

Axiom p1 (Efficiency) : If $(N; v) \in G^N$, it holds that,

$$\sum_{i \in N} f_i\,(N,v) = 1$$

Axiom p2 (Null Player) : If player $i \in N$ is null for $(N,v) \in G^N$, then,

$$f_i(N,v)(S) = 0$$

Axiom p3 (Symmetry): If $i,j \in N$ are two symmetric players in a game $(N,v) \in G^N$ then,

$$f_i(N,v)(S) = f_j(N,v)(S)$$

Axiom p4 (PGI-minimal monotonicity): If $(N,v), (N,w) \in G^N$, it holds that for all player $i \in N$ such that $M_i(v) \subseteq M_i(w)$,

IARJSET
ISSN (Online) 2393-8021
ISSN (Print) 2394-1588
International Advanced Research Journal in Science, Engineering and Technology
ISO 3297:2007 Certified
Vol. 4, Issue 9, September 2017

$$f_i(N,w)\sum_{j\in N} M_j(w) \geq f_i(N,v)\sum_{j\in N} M_j(v)$$

Considering the fact that $v(S) = 1$ if S is a winning coalition Axiom p1 can be expressed alternatively as follows:
Axiom p1' (Efficiency) : If $(N;\ v) \in G^N$, it holds that,

$$\sum_{i\in S} f_i\,(N,v) = v(S)$$

For remaining part of the paper we use Axiom p1' in place of Axiom p1. The following theorem ensures existence and uniqueness of the public good index

Theorem 2.1. There exist a unique Public Good Index f on G^N that satisfies Axiom p1-Axiom p4 and is given by,

$$f_i(N,v) = \frac{|M_i(v)|}{\sum_{j\in N}|M_j(v)|} \qquad (2.1)$$

In view of the Axiom p1' it can be rewritten as follows.
Theorem 2.2. There exist a unique Public Good Index f on G^N that satisfies Axiom p1'-Axiom p4 and is given by,

$$f_i(N,v)(S) = \frac{|M_i(v)|}{\sum_{j\in S}|M_j(v)|} \qquad (2.2)$$

In the remaining part of the paper we use (2.2) instead of (2.1)
Where $M_i(v)$ denotes the sets of minimal winning coalitions containing the player i in v.

2.2 FUZZY SIMPLE GAMES IN MULTILINEAR EXTENSION FORM

Define the fuzzy simple game $w_M: L(N) \rightarrow [0,1]$ in multilinear extension form as follows

$$w_M(A) = \sum_{T\in M(A^{\alpha}_{w_M})} \prod_{t\in T} \mu_A(T)v(T) \qquad (2.3)$$

Denote by $G^M_{FC}(N)$ the class of fuzzy simple games in multilinear extension form. For every$w_M \in G^M_{FC}(N)$we defne the function $f:\ G^M_{FC}(N) \rightarrow (\mathbb{R}^n)^{L(N)}$by

$$f^{\alpha}_i(w_M)(A) = \sum_{T\in M(A^{\alpha}_{w_{M_i}})} \prod_{t\in T} \mu_A(T)f^{'}_i(v)(T) \qquad (2.4)$$

Prior to defining the PGI for the class of fuzzy simple games we state and prove following results as follows.
Lemma 2.1. [4] Given $A \in L(N)$ and let $w_M \in G^m_{FC}(N)$ with $v \in G^N$being the associated simple game of w_M. Two players $i,j \in N$are α symmetric for win A iff $i,j \in N$aresymmetric for vin each $B_{ij}$where $B_{ij} = \{k \in N: \mu_B(i) < \alpha$ and $\mu_B(j) < \alpha$.
Proof. Proof is in the line of [4]
Lemma 2.2.[4] Given $A \in L(N)$ and $B \in L(N)$. Let $w_M \in G^m_{FC}(N)$ with $v \in G^N$being the associatedsimple game of w_M. Player $i \in N$is an α null player for $w_M$in A iff iis null for $v \in G^N$in each $T \in M(B \cup I)^{\alpha}_{w_{M_i}})$ with $\mu_B(j) > \alpha$when $j \neq i$, $\mu_B(j)>\alpha$ when $j = i$and $I \in L(N)$ such that $\mu_I(j)<\alpha$when $j \neq i$and $\mu_I(j)>\alpha$when $j = i$.
Proof. Proof in the line of [4]

3. FUZZY PGI FOR FUZZY SIMPLE GAMES

Definition 3.1.[4] A TU game w with fuzzy coalition [8] is said to be a fuzzy simple game,if for each fuzzy coalition $S \in L(N)$, we have $w(S) \in [0,1]$and $w$satisfy the following twoconditions.
1. $w(S_{\emptyset}) = 0, w(S_N) = 1$
2. Monotonicity: $w(S) \leq w(T)\ if\ S \subseteq T$
Let $G_{FC}(N)$ denote the class of all fuzzy simple games. Prior to the definition of public good index in fuzzy settings we define the following.

Definition 3.2.[4] Given $\alpha \in (0,1]$. If $S \in L(N)$and $w \in G_{FC}(N)$, the player $i \in N$is said to be αnull for $w$in Sif $w(T \cup I) = w(T)$for all $T \in L(S)$with $T(j) > \alpha$when $j \neq i$, $T(i) < \alpha$ and all $I \in L(N)$such that $I(i) > \alpha\ and\ I(j) < \alpha$when $j \neq i$.

Definition 3.3.[4] If $S \in L(N)$and $w \in G_{FC}(N)$, the players $i,j \in N$is said to be symmetric for win Sif $w(T \cup I) = w(T \cup J)$for all $T \in L(S)$with $T(i) = 0 = T(j)$and all $I,J \in L(N)$such that $I(i) = J(j)$and $I(j) = 0$when $j \neq i$and $J(i) = 0$when $i \neq j$.

UGC Approved Journal

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

ISO 3297:2007 Certified

Vol. 4, Issue 9, September 2017

In the following we define the fuzzy winning coalition and minimal fuzzy winning coalition as follows.

Definition 3.4.[4] Given $\alpha \in (0,1]$, let $S \in L(N)$and $w \in G_{FC}(N)$, a fuzzy coalition $T \in L(S)$is said to be α winning coalition if $w(T) \geq \alpha$ In analogy to this a fuzzy coalition $T\epsilon L(S)$is said to be loosing if $w(T) < \alpha$.

Definition 3.5.[4] Let $S\epsilon L(N)$and $w \in G_{FC}(N)$, an α winning coalition $T \in L(S)$is said to beα minimal winning coalition if every proper subset of $T \in L(S)$is a loosing coalition.
We denote the set of αwinning coalitions for win $S \in L(N)$ by $W(S_w^\alpha)$ and the setof α minimal winning coalitions by $M(S_w^\alpha)$. In the rest of the paper we denote the set ofminimal winning coalitions containing i by $M\left(S_{w_i}^\alpha\right)$. [4]

3.1 FUZZY PGI
Definition 3.6. A function $f : G_{FC}(N) \to (\mathbb{R}^n)^{L(N)}$is said to be a PGI in fuzzy settings in $G_{FC}(N)$if it satisfies the follwing axioms.
Axiom F1 (Efficiency): If $w \in G_{FC}(N)$ and $A \in L(N)$, then

$$\sum_{i \in N} f_i\, w(A) = w(A)$$

Axiom F2 (Null player Axiom): If player $i \in SuppA$is a null player for $w \in G_{FC}(N)$ in$A \in L(N)$ then

$$f_i(w)(A) = 0$$

Axiom F3 (Symmetry): For every pair of fuzzy symmetric player $i,j \in SuppA$and forevery $w \in G_{FC}(N)$ we must have

$$f_i(w)(A) = f_j(w)(A)$$

Axiom F4 (PGI-minimal monotonicity) : If $w, w' \in G_{FC}(N)$, it holds that

$$f_i^\alpha(w')(S) \sum_\alpha M\left(S_{(w_j')}^\alpha\right) \geq f_i^\alpha(w)(S) \sum_\alpha M(S_{w_j}^\alpha)$$

for all $\alpha \in (0,1]$ such that $M(A_{w_i}^\alpha) \sqsubseteq M(A_{w_i'}^\alpha)$

4. FUZZY PGI ON $G_{FC}^m(N)$

Theorem 4.1 The function $f : G_{FC}(N) \to (\mathbb{R}^n)^{l(N)}$given by (2.4) is the PGI for the class of fuzzy simple games in multilinear extension form given by(2.3) where$f'$is the associated crisp PGI.

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Proof. Axiom F1 (Efficiency): Let $A_\alpha = \{i_1, i_2, \ldots i_n\}$, where $m \leq n$.

$$f_{i_k}^\alpha(w_M)(A) = \sum_{T \in M\left(A_{w_{M_{i_k}}}^\alpha\right)} \prod_{i_j \in T} \mu_A(i_j) f_{i_k}'(v)(T)$$

$$\Rightarrow \sum_{k=1}^m f_{i_k}^\alpha(w_M)(A) = \sum_{k=1}^m \sum_{T \in M(A_{w_{M_{i_k}}}^\alpha)} \prod_{i_j \in T} \mu_A(i_j) f_{i_k}'(v)(T)$$

$$= \sum_{T \in M(A_{w_{M_{i_1}}}^\alpha)} \prod_{i_j \in T} \mu_A(i_j) f_{i_1}'(v)(T) + \cdots + \sum_{T \in M(A_{w_{M_{i_m}}}^\alpha)} \prod_{i_j \in T} \mu_A(i_j) f_{i_m}'(v)(T)$$

Let, $M\left(A_{w_M}^\alpha\right) = \{T_1, T_2, \ldots, T_p\}$. Hence,

$$\sum_{k=1}^m f_{i_k}^\alpha(w_M)(A) = \prod_{i_j \in T_1} \mu_A(i_j) \sum_{i_j \in T_1} f_{i_j}'(v)(T_1) + \cdots + \prod_{i_j \in T_p} \mu_A(i_j) \sum_{i_j \in T_p} f_{i_j}'(v)(T_p)$$

Let, $T_1 = \{i_1, i_2, \ldots, i_l\}$, where, $l \leq m$. Let the numbers of minimal coalition containing i_1 be m_1, i_2 be m_2,....,i_l be m_l. Hence

$$\sum_{i_j \in T_1} f_{i_j}'(v)(T_1) = \sum_{n=1}^l \frac{m_n}{m_1 + m_2 + \cdots m_l} = 1$$

IARJSET

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ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

ISO 3297:2007 Certified

Vol. 4, Issue 9, September 2017

Similarly,

$$\sum_{i_j \in T_p} f'_{i_j}(v)(T_p) = 1$$

Therefore,

$$\sum_{k=1}^{m} f_{i_k}^{\alpha}(w_M)(A) = \prod_{i_j \in T_1} \mu_A(i_j) + \cdots + \prod_{i_j \in T_p} \mu_A(i_j)$$

$$= w_M(A)$$

Axiom f2 (Null player Axiom): Null player follows from Lemma(2.2)
Axiom f3 (Symmetry): Symmetry follows from Lemma(2.1)
Axiom F4 (PGI-minimal monotonocity): Let $w_M, w'_M \in G_{FC}^m(N)$. Given that for al l$\alpha \in (0,1]$

$$M(A^{\alpha}_{(w_m)_i}) \subseteq M(A^{\alpha}_{(w'_M)_i}) \quad (4.8)$$

Hence

$$\sum_{\alpha} M(A^{\alpha}_{(w_m)_j}) \leq \sum_{\alpha} M(A^{\alpha}_{(w'_M)_j})$$

Following (4.2) and due to (4.8) we get

$$f_i^{\alpha}(w'_M(A) \geq f_i^{\alpha}(w_M(A)$$

Therefore,

$$f_i^{\alpha}(w'_M(A) \sum_{\alpha} M(A^{\alpha}_{(w'_M)_j}) \geq f_i^{\alpha}(w_M(A) \sum_{\alpha} M(A^{\alpha}_{(w_m)_j})$$

EXAMPLE

Let $N = \{1,2,3,4\}$. Let us consider an example, where $v: 2^N \to [0,1]$ is the simple game. Let Abe a fuzzy coalition over Ngiven by

$$A = \{<1,0.1>,<2,0.2>,<3,0.3>,<4,0.4>\}$$

Let, α be 0.1, then

$$W(A^{\alpha}_{w_M}) = \{\{2,4\},\{3,4\},\{2,3,4\}\}$$

Hence

$$M(A^{\alpha}_{w_M}) = \{\{2,4\},\{3,4\}\}$$

Thus using (2.3), $w_M(A) = 0.2$. After some computation a PGI, on $G_{FC}^m(N)$is obtained as (0, 0.03, 0.04, 0.13)

5. CONCLUSION

In this paper we introduce the notion of fuzzy PGI for fuzzy simple games in multilinear extension form. In future we will try to formulate more such power index.

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